On the critical dynamics of one-dimensional disordered Ising models

1987 J. Phys. A: Math. Gen. 20 L387

(http://iopscience.iop.org/0305-4470/20/6/009)

J. Phys. A: Math. Gen. **20** (1987) L387-L392. Printed in the UK

LETTER TO THE EDITOR

On the critical dynamics of one-dimensional disordered Ising models

M Droz†‡, J Kamphorst Leal da Silva†§, A Malaspinas†‡ and A L Stella||

† Department of Theoretical Physics, University of Geneva, 1211 Geneva 4, Switzerland
|| Dipartimento di Fisica ed Unita GNSM-CISM, Universita di Padova, 35131 Padova, Italy

Received 4 November 1986

Abstract. The critical dynamics of a disordered Ising ferromagnetic chain with two coupling constants ($J_1 \geqslant J_2 > 0$) is studied for Glauber dynamics. Using a domain wall argument the dynamical critical exponent z is found to be non-universal but independent of the disorder, namely $z = 1 + J_1/J_2$. The problem is formulated in terms of diffusion in a random medium. The diffusion is shown to be normal. Relationships with apparently very different diffusion problems, like the diffusion in hierarchically structured media, are established.

The critical dynamics of one-dimensional Ising models turned out recently to be richer than anticipated and several works have been devoted to it (Droz *et al* 1986a, b, Weir and Kosterlitz 1986, Kutasov *et al* 1986, Kamphorst 1986). In particular it has been shown that the critical dynamical exponent z, characterising the behaviour of the order parameter close to criticality, was not universal even for Glauber dynamics. For a ferromagnetic Ising chain with alternating couplings $J_1 \geqslant J_2 > 0$, it was shown (Droz *et al* 1986a) that $z = 1 + J_1/J_2$ for Glauber dynamics, a result to be compared with the value $z = 2$ for the uniform case. The same result could be reproduced by generalising a domain wall argument (DWA) due to Cordery *et al* (1981). Although the DWA is supposed to give an upper bound to the dynamical exponent z, it turns out that it predicts the exact values for all the one-dimensional cases for which an answer is known (Droz *et al* 1986a). Thus the value of z predicted by this method is probably exact in one dimension.

One of the key ingredients in the determination of z by a DWA is how the wall diffuses. For non-random systems the diffusion is usually normal. However even in this case, anomalous diffusion can be found in systems with hierarchical patterns of hopping rates (Maritan and Stella 1986a, b).

Recently a lot of attention has been devoted to the problem of diffusion in a random environment (Sinai 1982, Derrida and Pomeau 1982). It turns out that randomness can lead to quite anomalous diffusive behaviours. In view of this fact, one may suspect that the dynamics of a disordered version of the Ising chain with coupling constants taking values J_1 or J_2 randomly may differ from the one of the ordered chain.

‡ Work supported in part by the Swiss National Science Foundation.
§ On leave from Departamento de Fisica, Universidade de Minas Gerais, CP 702, 30000 Belo Horizonte, Minas Gerais, Brazil.

It is thus legitimate to study the critical dynamics of such a disordered chain. In this case there is no exact solution of the equations of motion. However, Dhar and Barma (1980) were able to give an analytical estimate of the time dependent magnetisation. By integrating their result one can have an upper bound for the dynamical exponent z.

The purpose of this letter is to consider this problem from the DWA point of view. The paper is organised as follows. First the model is defined and the DWA is briefly reviewed. The problem is then rephrased in terms of a random walk in a disordered medium. The corresponding equation for the mean passage time is solved and the result is averaged over the disorder. It turns out that the diffusion is normal and that the dynamical critical exponent is equal to $1+J_1/J_2$, as for the ordered case. Finally, the problem of diffusion in a random environment is considered from a different point of view. By a decimation transformation the original problem is mapped onto another apparently very different one, analogous to problems being recently considered in the literature like the diffusion in hierarchically structured media.

The model we consider is defined by the Hamiltonian

$$H=-\sum_i J_i s_i s_{i+1}$$

where the ferromagnetic couplings J_i are independent random variables. Although part of the discussion is quite general we will concentrate on a double-delta distribution for the couplings

$$P(J)=p\delta(J-J_1)+(1-p)\delta(J-J_2) \qquad J_1\geqslant J_2>0 \qquad 0<p<1. \tag{1}$$

The average correlation length $\bar{\xi}(T)$ describing the decay of the quenched average of the two point function is easily obtained as

$$\bar{\xi}^{-1}(T)=-\log[p\tanh(K_1)+(1-p)\tanh(K_2)]$$

where $K_i=J_i/k_BT$. For $T\to T_c=0$ we find

$$\bar{\xi}(T)\approx\exp(2K_2). \tag{2}$$

The dynamics is given by the usual master equation for the probability $p_t(\{s_i\})$ that the configuration $\{s_i\}$ is realised at time t, namely

$$\dot{p}_t(\{s_i\})=\sum_j[-\omega_j(s_j)p_t(s_1,\ldots,s_j\ldots)+\omega_j(-s_j)p_t(s_1,\ldots,-s_j\ldots)]. \tag{3}$$

The transition rates ω are partially determined by the detailed balance condition. For Glauber dynamics they are chosen as

$$\omega_j(s_j)=\tfrac{1}{2}\Gamma[1-\tfrac{1}{2}s_j(\gamma_j^+ s_{j-1}+\gamma_j^- s_{j+1})] \tag{4}$$

where

$$\gamma_n^\pm=\tanh(K_{n-1}+K_n)\pm\tanh(K_{n-1}-K_n). \tag{5}$$

From now on we will set $\Gamma=1$. For a temperature near T_c we define the dynamical critical exponent z by

$$\tau\approx\bar{\xi}^z$$

where τ is the relaxation time of the magnetisation averaged over the disorder.

Let us recall how z can be extracted by a domain wall argument in our case. For any given realisation of the distribution of the coupling constants, the behaviour of the relaxation time at low temperatures is determined by the time it takes for a domain wall (DW) to decay, i.e. the time it takes for a DW to cover a distance of the order of the correlation length. Generally speaking if one chooses the fastest possible mechanism for the DW motion the resulting value of z should be an upper bound to the exact one (Cordery *et al* 1981).

The statement of the DWA may be rephrased as follows. Given the DW at site $n=0$ at time $t=0$ what is the mean time it takes to decay at a distance ξ? In other words we look for the absorbtion time in a random walk with two absorbing barriers, one at ξ and the other at $-\xi$. The master equation corresponding to the motion of the DW is of the form

$$\dot{P}_n(t) = w^+_{n-1}P_{n-1}(t) + w^-_{n+1}P_{n+1}(t) - (w^+_n + w^-_n)P_n(t)$$

where

$$P_n(t) = P(n, t\,|\,0,0) \qquad P_n(0) = \delta_{n,0}. \tag{6}$$

Suppose now that the absorbing barriers are at A and $-A$. Let T_k be the mean time for the DW to be absorbed having started at k at time zero. Using the backward master equation one can derive the following equation for T_k (see, e.g., Gardiner 1983)

$$w^+_k(T_{k+1} - T_k) + w^-_k(T_{k-1} - T_k) = -1 \qquad -N = k \leqslant N$$

with the boundary conditions $T_{-A-1} = T_{A+1} = 0$, to express the absorbing barriers. These equations can be solved and one finds for $k=0$

$$T_0 = \frac{\sum_{l=-A-1}^{-1}\phi(l)\sum_{m=0}^{A}\phi(m)\sum_{n=-A}^{m}t_n - \sum_{l=0}^{A}\phi(l)\sum_{m=-A}^{-1}\phi(m)\sum_{n=-A}^{-1}t_n}{\sum_{n=-A-1}^{A}\phi(n)} \tag{7}$$

where

$$\phi(n) = \prod_{m=-A}^{n}\left(\frac{w^-_m}{w^+_m}\right) \qquad \phi(-A-1) = 1 \qquad t_n = (w^+_n\phi(n))^{-1}.$$

In our particular case one has, using (4),

$$\begin{aligned} w^-_{n+1} &= \tfrac{1}{2}(1 - \tanh(K_n - K_{n+1})) \\ w^+_n &= \tfrac{1}{2}(1 + \tanh(K_n - K_{n+1})). \end{aligned} \tag{8}$$

It follows easily that

$$\phi(n) = w^-_{-A}\,\mathrm{e}^{-2K_{-A}}\frac{\mathrm{e}^{2K_n}}{w^+_n}$$

and that

$$t(n) = \frac{\mathrm{e}^{2K_{-A}}}{w^-_{-A}}\,\mathrm{e}^{-2K_n}.$$

We can now average over the disorder. For A large enough the sum in the denominator of (7) simplifies by the corresponding sums in the numerator since they all tend to their average values. One can then average over the disorder. Replacing A by ξ, one finds

$$\tau = \bar{T}_0 \approx w_0\xi^2$$

with

$$w_0 = \overline{\left(\frac{\exp[-2(K_m - K_n)]}{1 + \tanh(K_m - K_{m+1})}\right)}.$$

For the double peaked distribution (2) one finds for low temperatures

$$w_0 \approx \exp(2K_1 - 2K_2).$$

Since

$$\bar{\xi} \approx \exp(2K_2)$$

we find

$$z = 1 + J_1/J_2.$$

This result is the same as for the Ising ferromagnet with alternate couplings discussed by Droz *et al* (1986a). On the other hand the same z may be extracted by integrating the time dependent magnetisation estimated for the same model by Dhar and Barma (1980). We note that also in their case this value of z represents an upper bound. The preceding calculation shows that the DW diffuses normally. This may look *a priori* surprising in view of known results on random walks with random transition rates (Sinai 1982, Derrida and Pomeau 1982) although the transition rates of the walk of the DW are not independent random variables as required in these references. In fact in our case the $\phi(n)$ are essentially constant. But it is exactly these objects that decide the behaviour of a random walk.

The dynamics of the domain wall described by (6) and (8) can be rephrased in the following way. One has a chain formed by a succession of 'weak' and 'strong' cells, distributed at random with the probability distribution given by (1). The hopping rates $W(i \to j)$ from cell i to cell j (strong or weak) are, in the low-temperature limit (see equation (8))

$$W(s \to s) = W(s \to w) = W(w \to w) = 1$$

and

$$W(w \to s) \equiv W = \exp(-2K_{\text{strong}} - 2K_{\text{weak}}).$$

We would like to relate the diffusion properties of this problem to other more 'standard' problems with randomness. A useful strategy to do that consists in making a decimation (or prefacing) transformation, mapping the original problem onto a new effective problem having the same asymptotic behaviour. The basic idea is the following. The clusters of strong cells constitute a sort of effective barrier between weak cells, with effective transition rates decreasing with the size of the cluster. The initial problem can be mapped onto one including only weak cells but with effective transition rates. This idea can be practically realised by performing a decimation of the strong cells. Decimation is typical of the renormalisation group approach to this type of problem (Rammal and Toulouse 1982, Khantha and Stinchcombe 1986).

The resulting diffusion problem for the weak cells is characterised by the following equation of motion for $\tilde{P}_i(\omega)$, the Laplace transform of the probability for the wall of being at time t in the weak cell i, having started at the weak cell 0 at time $t = 0$:

$$\beta_i \omega \tilde{P}_i(\omega) = \bar{W}_{i-1,i}[\tilde{P}_{i-1}(\omega) - \tilde{P}_i(\omega)] + \bar{W}_{i,i+1}[\tilde{P}_{i+1}(\omega) - \tilde{P}_i(\omega)] + \delta_{i,0} \tag{9}$$

where the Kronecker symbol expresses the initial condition. The hopping rates $\bar{W}_{i,i+1}$ are equal to 1 if the cells i and $i+1$ were not separated by strong cells before decimation. If on the other hand, they were separated by k strong cells, then

$$\bar{W}_{i,i+1} = \frac{W}{k+1} = \frac{\exp(-2K_{\text{strong}} - 2K_{\text{weak}})}{k+1} \qquad k = 1, 2, \ldots$$

The coefficients β_i are generated by the decimation transformation. Their physical origin lies in the fact that elimination of clusters of strong bonds implies waiting times for the cells adjacent to these clusters. These times are expected to grow with the length of the clusters. Indeed, one finds:

$\beta_i = 1$ if the cell i is surrounded by weak cells

$\beta_i = 1 + A(k)W$ if the cell i is adjacent to a cluster of $k-1$ strong cells ($k \geq 2$)

$\beta_i = 1 + [A(k_1) + A(k_2)]W$ if the cell i has a cluster of $k_1 - 1$ strong cells on one side and a cluster of $k_2 - 1$ strong cells on the other with

$$A(k) = \frac{(k-1)(2k-1)}{6k} \approx \frac{k}{3}$$

for large k.

Let us for the moment forget about the factors β_i entering in equation (9). In this case, the symmetry of the hopping rates $\bar{W}_{i,i+1}$ allows us to draw easily conclusions on the nature of the diffusion. Indeed our random model can be characterised as follows. The probability of having a transition rate $\bar{W}_{i,i+1} = 1$ is proportional to $1-p$ (see equation (2)). The probability of having a transition rate $\bar{W}_{i,i+1} = W/k$ is proportional to $p^{k-1}(1-p)$. Thus we are faced with a random hopping rate problem in which the probability of small hopping rates at each bond decreases like p^k.

Problems with random hopping rates continuously distributed, e.g. on the interval (0, 1) and taking values between ε and $\varepsilon + \mathrm{d}\varepsilon$ with probability proportional to $\rho(\varepsilon)\,\mathrm{d}\varepsilon$, have been studied extensively in the literature (Alexander *et al* 1981). It turns out that diffusion is normal provided the first negative moment of ρ converges. Also in our case an essentially continuous distribution is realised for small hopping rates ($k \to \infty$), and the corresponding first negative moment can be estimated as

$$\sum_{k=0}^{\infty} p^k \frac{k}{W} < \infty.$$

Thus, our random problem would definitely fall into the normal diffusion regime, i.e. it would belong to class a in the Alexander *et al* (1981) terminology. In order to get anomalous diffusion one would need a much stronger decay of the hopping rates for large k, e.g. like a power law $W_k \approx R^k$, ($R < 1$). In this case provided $R < p$ the behaviour would be anomalous and consistent with $\rho(\varepsilon) \approx \varepsilon^{-\alpha}$, with $\alpha = 1 - \log R/\log p$, i.e. it would fall into class c in the Alexander *et al* (1981) classification.

Let us now return to the full problem, including the factors β_i. Understanding the effect of these factors on the character of the diffusion is difficult for our random problem. We can however get an idea of what is going on by referring to what one

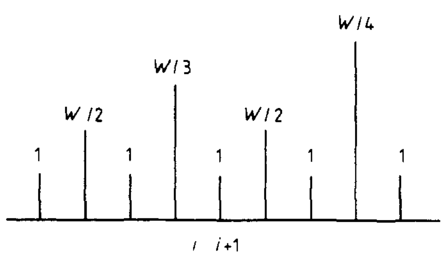

Figure 1. Ultrametric realisation of our model. The hopping rates between cells are distributed in a hierarchical way.

can call an ultrametric deterministic realisation of the random problem at hand. The random problems of the type discussed by Alexander *et al* have deterministic hierarchical counterparts having the same asymptotic properties (Teitel and Domany 1986, Maritan and Stella 1986a, b). We will restrict ourselves to $p = \frac{1}{2}$ for simplicity. The ultrametric realisation of our random problem would be the one depicted in figure 1. Each barrier bears specification of the corresponding rate. Notice that there is a fraction $(1-p) = \frac{1}{2}$ of barriers with rate 1 and a fraction $\approx p^k = (\frac{1}{2})^k$ of barriers with rate W/k. We can now consider the role played by the β_i. Assume that at each cell there is a waiting time factor $1 + A(k)W$ ($k \geq 2$ being the order of the high barrier near cell i, at the left or at the right). It is possible (Stella 1986) to discuss the effects of the β_i on the dynamics by renormalisation group arguments. One concludes that they do not affect the results obtained for $\beta_i = 1$ for all i, if $A(k)$ grows only linearly with k, as in our case. Thus the coincidence of asymptotic properties of the random problem and its hierarchical counterpart appears to hold also if the waiting time factors β_i are taken into account.

References

Alexander S, Bernasconi J, Schneider W R and Ohrbach R 1981 *Rev. Mod. Phys.* **53** 175

Cordery R, Sarker S and Tobochnik J 1981 *Phys. Rev.* B **24** 5402

Derrida B and Pomeau Y 1982 *Phys. Rev. Lett.* **48** 627

Dhar D and Barma J 1980 *J. Stat. Phys.* **22** 259

Droz M, Kamphorst Leal da Silva J and Malaspinas A 1986a *Phys. Lett.* **115A** 448

Droz M, Kamphorst Leal da Silva J, Malaspinas A and Yeomans J 1986b *J. Phys. A: Math. Gen.* **19** 2671

Gardiner C W (ed) 1983 *Handbook of Stochastic Methods* (Berlin: Springer)

Kamphorst Leal da Silva J 1986 *Phys. Lett.* **119A** 37

Khantha M and Stinchcombe R B 1986 *J. Phys. A: Math. Gen.* **20** 495

Kutasov D, Aharony A, Domany E and Kinzel W 1986 *Phys. Rev. Lett.* **56** 2229

Maritan A and Stella A L 1986a *J. Phys. A: Math. Gen.* **19** L259

—— 1986b *Phys. Rev. Lett.* **56** 1754

Rammal R and Toulouse G 1982 *Phys. Rev. Lett.* **49** 1982

Sinai Ya G 1982 *Theor. Prob. Appl.* **27** 247

Stella A L (ed) 1986 *Proc. Int. Meeting on Phase Transitions and Disordered Phenomena, Amalfi-Salerno, Italy* (Singapore: World Scientific) to be published

Teitel S and Domany E 1986 *Phys. Rev. Lett.* **56** 1755

Weir P O and Kosterlitz J M 1986 *Phys. Rev.* B **33** 391